

# WELL-POSEDNESS FOR FRACTIONAL NAVIER-STOKES EQUATIONS IN CRITICAL SPACES CLOSE TO $\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)$

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**ABSTRACT.** In this paper, we prove the well-posedness for the fractional Navier-Stokes equations in critical spaces  $G_n^{-(2\beta-1)}(\mathbb{R}^n)$  and  $BMO^{-(2\beta-1)}(\mathbb{R}^n)$ . Both of them are close to the largest critical space  $\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)$ . In  $G_n^{-(2\beta-1)}(\mathbb{R}^n)$ , we establish the well-posedness based on a priori estimates for the fractional Navier-Stokes equations in Besov spaces. To obtain the well-posedness in  $BMO^{-(2\beta-1)}(\mathbb{R}^n)$ , we find a relationship between  $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$  and  $BMO(\mathbb{R}^n)$  by giving an equivalent characterization of  $BMO^{-\zeta}(\mathbb{R}^n)$ .

## 1. INTRODUCTION

In this paper, we study the well-posedness of mild solutions to the fractional Navier-Stokes equations on the half-space  $\mathbb{R}_+^{1+n} = (0, \infty) \times \mathbb{R}^n$ ,  $n \geq 2$ :

$$(1.1) \quad \begin{cases} \partial_t u + (-\Delta)^\beta u + (u \cdot \nabla)u - \nabla p = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ u|_{t=0} = a, & \text{in } \mathbb{R}^n \end{cases}$$

with  $\beta \in (1/2, 1)$ . The mild solution to equations (1.1) is the fixed point of operator

$$(Tu)(t, x) = e^{-t(-\Delta)^\beta} a(x) - \int_0^t e^{-(t-s)(-\Delta)^\beta} P \nabla(u \otimes u)(s, x) ds.$$

Here

$$e^{-t(-\Delta)^\beta} f(x) := K_t^\beta(x) * f(x) \quad \text{with } \widehat{K_t^\beta}(\xi) = e^{-t|\xi|^{2\beta}}$$

and  $P$  is the Helmholtz-Weyl projection:

$$P = \{P_{j,k}\}_{j,k=1,\dots,n} = \{\delta_{j,k} + R_j R_k\}_{j,k=1,\dots,n}$$

with the Kronecker symbol  $\delta_{j,k}$  and the Riesz transform  $R_j = \partial_j(-\Delta)^{-1/2}$ .

Note that the following scaling

$$(1.2) \quad u_\lambda(t, x) = \lambda^{2\beta-1} u(\lambda^{2\beta} t, \lambda x), \quad p_\lambda(t, x) = \lambda^{4\beta-2} p(\lambda^{2\beta} t, \lambda x), \quad a_\lambda(x) = \lambda^{2\beta-1} a(\lambda x)$$

is important for equations (1.1). This leads us to study equations (1.1) in critical function spaces which are invariant under the scaling  $f(x) \longrightarrow \lambda^{2\beta-1} f(\lambda x)$ .

When  $\beta = 1$ , equations (1.1) become the classical Navier-Stokes equations. The existence of mild solutions has been established locally in time and global for small initial data in various critical spaces. Especially, Koch and Tataru in [14] proved the well-posedness of classical Navier-Stokes equations in the space  $BMO^{-1}(\mathbb{R}^n) =$

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$\nabla \cdot (BMO(\mathbb{R}^n))^n$ . Xiao in [26] generalized the results of Koch and Tataru [14] to  $Q_{\alpha;\infty}^{-1}(\mathbb{R}^n)$  for  $\alpha \in (0, 1)$ . Chen and Xin in [5] studied the classical Navier-Stokes equations in several critical spaces. See, Kato [13], Cannone [3], Giga and Miyakawa [10], Bourgain and Pavlović [2] and the references therein for more history and recent development.

For general case, Lions [17] proved the global existence of the classical solutions to equations (1.1) when  $\beta \geq \frac{5}{4}$  in dimensional 3. Wu in [21] obtained similar result for  $\beta \geq \frac{1}{2} + \frac{n}{4}$  in dimension  $n$ . For the important case  $\beta < \frac{1}{2} + \frac{n}{4}$ , Wu in [22]-[23] considered the existence of solution to equations (1.1) in  $\dot{B}_{p,q}^{1+\frac{n}{p}-2\beta}(\mathbb{R}^n)$ . In Li and Zhai [15]-[16], inspired by Koch and Tataru [14] and Xiao [26], they studied equations (1.1) in critical space  $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n) = \nabla \cdot (Q_{\alpha}^{\beta}(\mathbb{R}^n))^n$  for  $\beta \in (1/2, 1)$  and  $\alpha \in (0, \beta)$ . Here  $Q_{\alpha}^{\beta}(\mathbb{R}^n)$  for  $\alpha \in (-\infty, \beta)$  is the set of all measurable functions with

$$\sup_I (l(I))^{2(\alpha+\beta-1)-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2(\alpha-\beta+1)}} dx dy < \infty$$

where the supremum is taken over all cubes  $I$  with the edge length  $l(I)$  and the edges parallel to the coordinate axes in  $\mathbb{R}^n$ .  $Q_{\alpha}^{\beta}(\mathbb{R}^n)$  is a generalization of  $Q_{\alpha}(\mathbb{R}^n)$  studied by Essen, Janson, Peng and Xiao [8], Xiao [25], Dafni and Xiao [6]-[7]. Meanwhile, Li and Zhai [15] proved that Besov space  $\dot{B}_{\infty,\infty}^{1-2\beta}(\mathbb{R}^n)$  for  $\beta \in (1/2, 1)$  is the biggest one among the critical spaces of equations (1.1).

In this paper, we accomplish two major goals. First, we prove the well-posedness for equations (1.1) in spaces  $G_n^{-(2\beta-1)}(\mathbb{R}^n)$  for  $\beta \in (1/2, 1)$ . Here, for  $s > 0$ ,

$$G_p^{-s}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : |f| \in \mathcal{S}'(\mathbb{R}^n), \|f\|_{G_p^{-s}(\mathbb{R}^n)} = \sup_{t>0} t^{\frac{sn}{2p\beta}} \|e^{-t(-\Delta)^{\beta}} |f|\|_{L^{\infty}(\mathbb{R}^n)} < \infty \right\}.$$

Second, to obtain the well-posedness in  $BMO^{-(2\beta-1)}(\mathbb{R}^n)$  for  $\beta \in (1/2, 1)$ , we find a relation between  $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$  and  $BMO(\mathbb{R}^n)$ :

$$(1.3) \quad Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n) = (-\Delta)^{\frac{2\beta-1}{2}} BMO(\mathbb{R}^n) = BMO^{-(2\beta-1)}(\mathbb{R}^n)$$

for  $\alpha = 1 - \beta$  and  $\beta \in (1/2, 1)$ , by giving an equivalent characterization of  $BMO^{-\zeta}(\mathbb{R}^n)$ . Our well-posedness results extend that of Chen and Xin [5], Koch and Tataru [14]. The relation (1.3) between  $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$  for  $\beta \in (1/2, 1)$  and  $BMO(\mathbb{R}^n)$  gives us a clear link between  $Q_{\alpha}^{\beta}(\mathbb{R}^n)$  and  $BMO(\mathbb{R}^n)$ . When  $\alpha \neq 1 - \beta$ , an interesting problem is whether or not there is a similar link between  $Q_{\alpha}^{\beta}(\mathbb{R}^n)$  and  $BMO(\mathbb{R}^n)$ .

The space  $BMO^{-\zeta}(\mathbb{R}^n)$  was introduced by Zhou and Gala in [28] by using heat semigroup  $e^{t\Delta}$ . In the following, we define  $BMO^{-\zeta}(\mathbb{R}^n)$  by  $e^{-t(-\Delta)^{\beta}}$  for  $\beta \in (1/2, 1)$ . This is motivated by the following well-known facts.

For a  $C^{\infty}$  real-valued function on  $\mathbb{R}^n$  satisfying the properties:

$$(1.4) \quad \phi_j \in L^1(\mathbb{R}^n), |\phi_j(x)| \lesssim (1+|x|)^{-(n+1)}, \int_{\mathbb{R}^n} \phi_j(x) dx = 0 \text{ and } (\phi_j)_t(x) = t^{-n} \phi_j\left(\frac{x}{t}\right),$$

$$(1.5) \quad f \in BMO(\mathbb{R}^n) \iff \sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{-n} \int_0^r \int_{|y-x|<r} |f * \phi_t(y)|^2 t^{-1} dt dy < \infty.$$

Here  $A \lesssim B$  means  $A \leq CB$  with  $C > 0$ . Thus  $BMO(\mathbb{R}^n)$  can be defined equivalently as

$$(1.6) \quad \|f\|_{BMO(\mathbb{R}^n)}^2 = \sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{-n} \int_0^{r^{2\beta}} \int_{|y-x| < r} |\nabla e^{-t(-\Delta)^\beta} f(y)|^2 t^{\frac{1-\beta}{\beta}} dt dy < \infty.$$

Then, (1.6) leads us to introduce  $BMO^{-\zeta}(\mathbb{R}^n)$  as follows.

**Definition 1.1.** For  $\beta \in (1/2, 1)$ ,  $0 \leq \zeta \leq n/2$ , define  $BMO^{-\zeta}(\mathbb{R}^n)$  as the set of all measurable functions  $f$  with

$$\|f\|_{BMO^{-\zeta}(\mathbb{R}^n)}^2 = \sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{-n} \int_0^{r^{2\beta}} \int_{|y-x| < r} t^{\frac{\zeta-\beta}{\beta}} |e^{-t(-\Delta)^\beta} f(y)|^2 dt dy < \infty.$$

*Remark 1.2.* Obviously,  $BMO^{-\zeta}(\mathbb{R}^n)$  is invariant under the scaling  $f(x) \rightarrow \lambda^\zeta f(\lambda x)$ . Note that  $Q_{\alpha; \infty}^{\beta, -1}(\mathbb{R}^n)$  is invariant under the scaling  $f(x) \rightarrow \lambda^{2\beta-1} f(\lambda x)$ . Thus  $BMO^{-\zeta}(\mathbb{R}^n)$  will be more useful than  $Q_{\alpha; \infty}^{\beta, -1}(\mathbb{R}^n)$ .

We state our main results as follows. The first one is a priori estimates in homogeneous Besov spaces for the fractional Navier-Stokes equations.

**Proposition 1.3.** Let  $2 - 2\beta < w < 2\beta$ ,  $1 + n/p + w < 4\beta$ ,  $2 \leq n \leq p \leq \infty$ ,  $1 \leq q \leq \infty$  and

$$a \in (\mathcal{S}'(\mathbb{R}^n))^n, f(t) \in (\dot{B}_{p, \infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n))^{n \times n}.$$

Then the solution to the integral equation

$$u(t) = e^{-t(-\Delta)^\beta} a + \int_0^t e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot f(s) ds$$

satisfies the estimates

$$\|u(t)\|_{\dot{B}_{p, q}^{-1+\frac{n}{p}}(\mathbb{R}^n)} \lesssim \|a\|_{\dot{B}_{p, q}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w}{2\beta}+1-\frac{1}{\beta}} \|f(s)\|_{\dot{B}_{p, \infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)}$$

and

$$t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{p, \infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \lesssim \|a\|_{\dot{B}_{p, \infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p, \infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)}$$

provided the right-hand sides of the above inequalities are finite, respectively.

Applying Proposition 1.3, we obtain the existence of solution to equations (1.1).

**Theorem 1.4.** Let  $n \geq 2$ ,  $\beta \in (1/2, 1)$ ,  $\max\{2\beta-1, 2-2\beta\} < w < 2\beta$ ,  $1+n/p+w < 4\beta$ ,  $a \in G_n^{-(2\beta-1)}(\mathbb{R}^n)$ ,  $\nabla \cdot a = 0$ . If  $\|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)}$  is small enough, then there is a unique solution to (1.1) satisfying

$$\|u(t)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + t^{\frac{1}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{\infty, \infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \lesssim \|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)}.$$

Similar to Theorem 1.4, we can prove the existence of solutions to the fractional magnetohydrodynamics equations

$$(1.7) \quad \begin{cases} \partial_t u + (-\Delta)^\beta u + u \cdot \nabla u + \nabla p - b \cdot \nabla b = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ \partial_t b + (-\Delta)^\beta b + b \cdot \nabla b - b \cdot \nabla u = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ \nabla \cdot u = \nabla \cdot b = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0, & \text{in } \mathbb{R}^n. \end{cases}$$

We refer the readers to Wu [21] and [24] and the references therein for more information about this system.

**Theorem 1.5.** *Let  $n \geq 2$ ,  $\beta \in (1/2, 1)$ ,  $\max\{2\beta-1, 2-2\beta\} < w < 2\beta$ ,  $1+n/p+w < 4\beta$ ,  $(u_0, b_0) \in G_n^{-(2\beta-1)}(\mathbb{R}^n)$ ,  $\nabla \cdot u_0 = 0$  and  $\nabla \cdot b_0 = 0$ . If  $\|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|b_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)}$  is small enough, then there is a unique mild solution to (1.7) satisfying*

$$\begin{aligned} \|u(t)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + t^{\frac{1}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} &\lesssim \|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)}, \\ \|b(t)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + t^{\frac{1}{2\beta}} \|b(t)\|_{L^\infty(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|b(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} &\lesssim \|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)}. \end{aligned}$$

Using Proposition 1.3, we get the existence of solutions to equations (1.1) in  $\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)$ .

**Proposition 1.6.** *Let  $n \geq 2$ ,  $\beta \in (1/2, 1)$ ,  $n \leq p < \infty$ ,  $\max\{2\beta - n/p, 2 - 2\beta\} < w < 2\beta$  and  $1+n/p+w < 4\beta$ . Assume that  $a \in (\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n))^n$  and  $\nabla \cdot a = 0$ . If  $\|a\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)}$  is small enough, then there exists a unique solution to equations (1.1) satisfying*

$$\|u(t)\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + t^{\frac{2\beta-1}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \lesssim \|a\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)}.$$

*Remark 1.7.* In [23], Wu established a result similar to Proposition 1.6 by using lower bounds for the integral involving  $(-\Delta)^\beta$ .

Now, we study the properties of  $BMO^{-\zeta}(\mathbb{R}^n)$ .

**Proposition 1.8.** ( *$BMO^{-\zeta}$  and Besov spaces*) *Let  $\beta \in (1/2, 1)$ . For any  $f \in \mathcal{S}'(\mathbb{R}^n)$  and  $t > 0$ , we have*

$$r^\zeta \|e^{-r^{2\beta}(-\Delta)^\beta} f\|_{L^\infty} \lesssim \left( r^{-n} \int_0^{r^{2\beta}} \int_{|x-x_0| \leq s} s^{\zeta-1+\frac{1-\beta}{\beta}} |e^{-s(-\Delta)^\beta} f(x)|^2 dx ds \right)^{1/2},$$

that is,  $BMO^{-\zeta}(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^{-\zeta}(\mathbb{R}^n)$ .

**Proposition 1.9.** *A distribution  $f$  belongs to  $BMO^{-\zeta}(\mathbb{R}^n)$  if and only if there exists a distribution  $g \in BMO(\mathbb{R}^n)$  such that  $f = (-\Delta)^{\frac{\zeta}{2}} g$ .*

*Remark 1.10.* (i) Zhou and Gala established results similar to Propositions 1.8-1.9 for  $BMO^{-\zeta}(\mathbb{R}^n)$  defined by heat semigroup  $e^{t\Delta}$ . Thus,  $BMO^{-\zeta}(\mathbb{R}^n)$  is independent of  $e^{-t(-\Delta)^\beta}$  for  $\beta \in (1/2, 1]$ .

(ii) It follows from the definition of  $BMO^{-\zeta}(\mathbb{R}^n)$  and  $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$  (see [15]) that when  $\alpha = 1 - \beta$ ,  $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n) = BMO^{-\zeta}(\mathbb{R}^n)$  for  $\zeta = 2\beta - 1$ . Thus, we can obtain the existence of mild solution to equations (1.1) with initial data in  $BMO^{-(2\beta-1)}(\mathbb{R}^n)$  as follows.

We need to define some notations.

**Definition 1.11.** Let  $1/2 < \beta < 1$ .

(i) A tempered distribution  $f$  on  $\mathbb{R}^n$  belongs to  $BMO_T^{-(2\beta-1)}(\mathbb{R}^n)$  provided

$$\|f\|_{BMO_T^{-(2\beta-1)}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r \in (0, T)} \left( r^{-n} \int_0^{r^{2\beta}} \int_{|y-x| < r} |K_t^\beta * f(y)|^2 t^{\frac{\beta-1}{\beta}} dy dt \right)^{1/2} < \infty;$$

- (ii) A tempered distribution  $f$  on  $\mathbb{R}^n$  belongs to  $\overline{VMO^{-(2\beta-1)}}(\mathbb{R}^n)$  provided  $\lim_{T \rightarrow 0} \|f\|_{BMO_T^{-(2\beta-1)}(\mathbb{R}^n)} = 0$ ;
- (iii) A function  $g$  on  $\mathbb{R}_+^{1+n}$  belongs to the space  $X_T^\beta(\mathbb{R}^n)$  provided

$$\begin{aligned} \|g\|_{X_T^\beta(\mathbb{R}^n)} &= \sup_{t \in (0, T)} t^{1-\frac{1}{2\beta}} \|g(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \\ &+ \sup_{x \in \mathbb{R}^n, r^{2\beta} \in (0, T)} \left( r^{-n} \int_0^{r^{2\beta}} \int_{|y-x| < r} |g(t, y)|^2 t^{\frac{\beta-1}{\beta}} dy dt \right)^{1/2} < \infty. \end{aligned}$$

**Proposition 1.12.** [15] *Let  $n \geq 2$ ,  $1/2 < \beta < 1$ . Then*

- (i) *The fractional Navier-Stokes system (1.1) has a unique small global mild solution in  $(X_\infty^\beta)^n$  for all initial data  $a$  with  $\nabla \cdot a = 0$  and  $\|a\|_{(BMO_\infty^{-(2\beta-1)})^n}$  being small.*
- (ii) *For any  $T \in (0, \infty)$  there is an  $\varepsilon > 0$  such that the fractional Navier-Stokes system (1.1) has a unique small mild solution in  $(X_T^\beta)^n$  on  $(0, T) \times \mathbb{R}^n$  when the initial data  $a$  satisfies  $\nabla \cdot a = 0$  and  $\|a\|_{(BMO_T^{-(2\beta-1)})^n} \leq \varepsilon$ . In particular for all  $a \in \overline{(VMO^{-(2\beta-1)})^n}$  with  $\nabla \cdot a = 0$  there exists a unique small local mild solution in  $(X_T^\beta)^n$  on  $(0, T) \times \mathbb{R}^n$ .*

*Remark 1.13.* (i)  $G_n^{-(2\beta-1)}(\mathbb{R}^n)$  and  $BMO^{-(2\beta-1)}(\mathbb{R}^n)$  are different critical spaces for equations (1.1) and no inclusion relation between them.

(ii) Proposition 1.12 is an generalization of Koch and Tataru [14, Theorem 2-3] since  $BMO^{-(2\beta-1)}(\mathbb{R}^n) = (-\Delta)^{-\frac{2\beta-1}{2}} BMO(\mathbb{R}^n)$ .

(iii) Similar to Proposition 1.12, we can consider the well-posedness for dissipative quasi-geostrophic equations in  $BMO^{-(2\beta-1)}(\mathbb{R}^2)$ .

The rest of this paper is organized as follows. In Section 2, we give the definition and some basic properties of Besov spaces. In Section 3, we prove Proposition 1.3. In Section 4, we verify Theorem 1.4 based on a prior estimates for fractional Navier-Stokes equations. In Section 5, we show Theorem 1.5 by the contraction mapping principle. In Section 6, we demonstrate Proposition 1.6 by applying the contraction mapping principle and a prior estimates for fractional Navier-Stokes equations. In final two section, we establish Propositions 1.8 and 1.9.

## 2. PRELIMINARY LEMMAS

In this section, we provide the definition and several properties of the homogeneous Besov spaces.

We recall the definition of homogeneous Besov spaces. For details, see Berg and Lofstrom [1] and Triebel [19]-[20]. We start with the Fourier transform. The Fourier transform  $\widehat{f}$  of  $f \in \mathcal{S}$  is defined as

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-x \cdot \xi} dx.$$

Here  $\mathcal{S}(\mathbb{R}^n)$  denotes the Schwartz class of rapidly decreasing smooth functions and  $\mathcal{S}'(\mathbb{R}^n)$  is the space of tempered distributions. The fractional power of the Laplacian can be defined by the Fourier transform. For  $\theta \in \mathbb{R}$ ,

$$(-\Delta)^{\theta/2} f(\xi) = |\xi|^\theta \widehat{f}(\xi).$$

We will use  $f^\vee$  to denote the inverse Fourier transform of  $f$ . Then we introduce the Littlewood-Paley decomposition by means of  $\{\varphi_j\}_{j=-\infty}^\infty$ . Take a function  $\phi \in C_0^\infty$  with  $\text{supp}(\phi) = \{\xi \in \mathbb{R}^n : 1/2 < |\xi| \leq 2\}$  such that  $\sum_{j=-\infty}^\infty \phi(2^{-j}\xi) = 1$  for all  $\xi \neq 0$ . Then we define functions  $\varphi_j (j = 0, \pm 1, \pm 2, \dots)$  as

$$\widehat{\varphi_j}(\xi) = \phi(2^{-j}\xi).$$

Let  $\Delta_j f = \varphi_j * f$ , for  $j = 0, \pm 1, \pm 2, \pm 3, \dots$ . Then, for  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , we define

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} = \left( \sum_{j=-\infty}^\infty (2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R}^n)})^q \right)^{1/q}, \quad 1 \leq q < \infty$$

$$\|f\|_{\dot{B}_{p,\infty}^s(\mathbb{R}^n)} = \sup_{-\infty < j < \infty} (2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R}^n)}), \quad q = \infty,$$

where  $L^p(\mathbb{R}^n)$  means the usual Lebesgue space on  $\mathbb{R}^n$  with the norm  $\|\cdot\|_{L^p(\mathbb{R}^n)}$ . The homogeneous Besov space  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  is defined by

$$\dot{B}_{p,q}^s(\mathbb{R}^n) = \{f \in \mathcal{S}' : \|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} < \infty\}.$$

We will use the following properties about homogeneous Besov space.

**Lemma 2.1.** *The following properties hold:*

- (i) If  $1 \leq q_1 \leq q_2 \leq \infty$ ,  $1 \leq p \leq \infty$  and  $s \in \mathbb{R}$ , then  $\dot{B}_{p,q_1}^s(\mathbb{R}^n) \hookrightarrow \dot{B}_{p,q_2}^s(\mathbb{R}^n)$ .
- (ii) If  $1 \leq p_1 \leq p_2 \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $-\infty < s_1 \leq s_2 < \infty$  and  $s_2 - \frac{n}{p_2} = s_1 - \frac{n}{p_1}$ , then

$$\dot{B}_{p_2,q}^{s_2}(\mathbb{R}^n) \hookrightarrow \dot{B}_{p_1,q}^{s_1}(\mathbb{R}^n).$$

- (iii) If  $\beta, s \in \mathbb{R}$ ,  $1 \leq p, q \leq \infty$ , then the operator  $(-\Delta)^{\beta/2}$  is an isomorphism from  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  to  $\dot{B}_{p,q}^{s-\beta}(\mathbb{R}^n)$ .

**Lemma 2.2.** *Let  $0 < \theta < 1$ ,  $1 \leq p, q \leq \infty$ ,  $-\infty < s_1 < s_2 < \infty$  and  $s = (1 - \theta)s_1 + \theta s_2$ . Then*

$$(\dot{B}_{p,\infty}^{s_1}(\mathbb{R}^n), \dot{B}_{p,\infty}^{s_2}(\mathbb{R}^n))_{\theta,q} = \dot{B}_{p,q}^s(\mathbb{R}^n)$$

for  $s = s_1(1 - \theta) + s_2\theta$ , where  $(\cdot, \cdot)_{\theta,q}$  means the real interpolation functor, see Berg and Lofstrom [1].

We will use the  $L^p - L^q$ -type estimates for  $e^{-t(-\Delta)^\theta}$  in homogeneous Besov spaces. For  $\theta = 1$ , the  $L^p - L^p$ -estimates for  $e^{t\Delta}$  in Besov spaces were studied by Kozono, Ogawa and Taniuchi in [12]. Zhai in [27] proved the general case of  $\theta > 0$ .

**Lemma 2.3.** *Let  $\theta > 0$  and  $\zeta \geq 0$ . If  $s_1 \leq s_2$ ,  $1 \leq p_1 \leq p_2 \leq \infty$  and  $1 \leq q \leq \infty$ , then*

$$(2.1) \quad \|e^{-t(-\Delta)^\theta} f\|_{\dot{B}_{p_2,q}^{s_2}(\mathbb{R}^n)} \lesssim t^{-\frac{s_2-s_1}{2\theta} - \frac{n}{2\theta}(\frac{1}{p_1} - \frac{1}{p_2})} \|f\|_{\dot{B}_{p_1,q}^{s_1}(\mathbb{R}^n)}.$$

The following equivalent characterization of homogeneous Besov spaces will be useful.

**Lemma 2.4.** ([19]) *Let  $0 < s < 1$  and  $1 \leq p \leq \infty$ , then in  $\dot{B}_{p,\infty}^s(\mathbb{R}^n)$ , we have*

$$\|f\|_{\dot{B}_{p,\infty}^s(\mathbb{R}^n)} \equiv \sup_{y \neq 0} \frac{\|f(\cdot + y) - u(\cdot)\|_{L^p(\mathbb{R}^n)}}{|y|^s}.$$

We need a variant of Mikhlin theorem on Fourier multipliers.

**Lemma 2.5.** ([19]) *Let  $-\infty < s < \infty$  and  $\phi(x)$  be a complex-valued infinitely differentiable function on  $\mathbb{R}^n \setminus \{0\}$  so that*

$$\sup_{j \leq k} \sup_{x \in \mathbb{R}^n} |x|^j |\nabla^j \phi(x)| < \infty$$

*for a sufficiently large positive integer  $k$ . Then*

$$\|(\phi \hat{u})^\vee\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}$$

*for  $u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$  with  $1 \leq p, q \leq \infty$ .*

We need a useful lemma, see for example, Grafakos [11], Frazier, Jawerth and Weiss [9].

**Lemma 2.6.** *Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ . Then the following statements are equivalent:*

- (i)  $f \in BMO(\mathbb{R}^n)$ ;
- (ii) *for all  $\phi \in \mathcal{S}'(\mathbb{R}^n)$  satisfying:*

$$\int_{\mathbb{R}^n} \phi(x) dx = 0, \quad \sup_{\xi \in \mathbb{R}^n} \int_0^\infty |\hat{\phi}(t\xi)|^2 \frac{dt d\xi}{t} < \infty$$

*and  $|\phi(x)| \lesssim \frac{1}{(1+|x|)^{n+c}}$  for some  $c$ , then the measure*

$$d\mu(t, x) = |\phi_t * b(x)|^2 \frac{dt dx}{t}$$

*is a Carleson measure on  $\mathbb{R}_+^{1+n}$ .*

**Lemma 2.7.** *Let  $2\beta - 1 < w < 2\beta$ ,  $2 \leq n \leq p \leq \infty$ ,  $1 \leq q \leq \infty$ , then we have*

$$\begin{aligned} & t^{\frac{2\beta-1}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + t^{\frac{w-(2\beta-1)}{2\beta}} \|u(t)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\ & \lesssim \|u(t)\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)}. \end{aligned}$$

*Proof.* It follows from Lemmas 2.1-2.2 and [19, Proposition 2.5.7] that

$$\dot{B}_{p,\infty}^{w-2\beta}(\mathbb{R}^n) = \left( \dot{B}_{p,\infty}^{-(2\beta-1)}(\mathbb{R}^n), \dot{B}_{p,\infty}^{w-(2\beta-1)}(\mathbb{R}^n) \right)_{\frac{w-1}{w}, \infty}$$

and

$$\dot{B}_{\infty,\infty}^0(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n) \supset \dot{B}_{\infty,1}^0(\mathbb{R}^n) = \left( \dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n), \dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n) \right)_{\frac{2\beta-1}{w}, 1}$$

which contains

$$\left( \dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n), \dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n) \right)_{\frac{2\beta-1}{w}, 1}.$$

Hence, we can get

$$\begin{aligned} & \|u(t, t^{\frac{1}{2\beta}} \cdot)\|_{L^\infty(\mathbb{R}^n)} + \|u(t, t^{\frac{1}{2\beta}} \cdot)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\ & \lesssim \|u(t, t^{\frac{1}{2\beta}} \cdot)\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \|u(t, t^{\frac{1}{2\beta}} \cdot)\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)}. \end{aligned}$$

By changing variables, we can find that

$$\begin{aligned} & t^{\frac{2\beta-1}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + t^{\frac{w-(2\beta-1)}{2\beta}} \|u(t)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\ & \lesssim \|u(t)\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)}. \end{aligned}$$

□

**Lemma 2.8.** For  $\beta \in (1/2, 1)$ ,  $u, v \in (L^\infty(\mathbb{R}^n))^n \cap (G_n^{-(2\beta-1)}(\mathbb{R}^n))^n$ , then we have

$$\|e^{-t(-\Delta)^\beta} P \nabla \cdot (u \otimes v)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \lesssim t^{\frac{1-2\beta}{2\beta}} \|u\|_{L^\infty(\mathbb{R}^n)} \|v\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)}.$$

*Proof.* It is easy to see that for  $\beta \in (1/2, 1)$ ,

$$\|\partial_{x_i} \partial_{x_j} \Delta^{-1} \partial_{x_k} K_t^\beta(x)\|_{L^1(\mathbb{R}^n)} \lesssim t^{-\frac{1}{2\beta}}(i, j, k = 1, 2, \dots, n).$$

Since the operation with respect to the convolution is commutative, by letting

$$K_{i,j,k,t} = (\delta_{ij} - \partial_{x_i} \partial_{x_j} \Delta^{-1}) \partial_{x_k} K_t^\beta(x),$$

we have, for  $s > 0$

$$\begin{aligned} & \|e^{-s(-\Delta)^\beta} |e^{-t(-\Delta)^\beta} P \nabla \cdot (u \otimes v)|\|_{L^\infty(\mathbb{R}^n)} \\ & \leq \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \|K_s^\beta * |K_{i,j,k,t} * (u_k v_j)|\|_{L^\infty(\mathbb{R}^n)} \\ & \leq \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \|K_s^\beta * |K_{i,j,k,t}| * |u_k v_j|\|_{L^\infty(\mathbb{R}^n)} \\ & \leq \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \| |K_{i,j,k,t} * K_s^\beta * |u_k v_j| \|_{L^\infty(\mathbb{R}^n)} \\ & \leq \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \|K_{i,j,k,t}\|_{L^1(\mathbb{R}^n)} \|K_s^\beta * |u_k v_j|\|_{L^\infty(\mathbb{R}^n)} \\ & \leq \left( \|\nabla K_t^\beta\|_{L^1(\mathbb{R}^n)} + \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \|\partial_{x_i} \partial_{x_j} \Delta^{-1} \partial_{x_k} K_t^\beta\|_{L^1(\mathbb{R}^n)} \right) \|e^{-s(-\Delta)^\beta} |u_k v_j|\|_{L^\infty(\mathbb{R}^n)} \\ & \lesssim t^{-\frac{1}{2\beta}} \|u\|_{L^\infty(\mathbb{R}^n)} \|e^{-s(-\Delta)^\beta} |v|\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

Thus, we get

$$\begin{aligned} & \sup_{s>0} s^{\frac{2\beta-1}{2\beta}} \|e^{-s(-\Delta)^\beta} |e^{-t(-\Delta)^\beta} P \nabla \cdot (u \otimes v)|\|_{L^\infty(\mathbb{R}^n)} \\ & \lesssim t^{-\frac{1}{2\beta}} \|u\|_{L^\infty(\mathbb{R}^n)} \sup_{s>0} s^{\frac{2\beta-1}{2\beta}} \|e^{-s(-\Delta)^\beta} |v|\|_{L^\infty(\mathbb{R}^n)} \end{aligned}$$

and finishes the proof. □

### 3. PROOF OF PROPOSITION 1.3

It follows from Lemma 2.5 that

$$(3.1) \quad \|Pv\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} + \|\nabla(-\Delta)^{-1/2} v\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \lesssim \|v\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}.$$

On the other hand, it is easy to see that for  $k \geq 0$ ,

$$\|\nabla^k e^{-t(-\Delta)^\beta} v\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{k}{2\beta}} \|v\|_{L^p(\mathbb{R}^n)}.$$



Then (iii) of Lemma 2.1 tells us

$$\begin{aligned}
& \|u(t) - e^{-t(-\Delta)^\beta} a\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \|\Delta^{\beta+1} \int_0^t e^{-(t-s+\tau)(-\Delta)^\beta} \Delta^{-1} P \nabla \cdot f(s) ds\|_{L^p(\mathbb{R}^n)} \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \int_0^t (t+\tau-s)^{-\frac{2\beta}{2\beta}} \|\Delta e^{-\frac{(t+\tau-s)}{2}(-\Delta)^\beta} \Delta^{-1} P \nabla \cdot f(s)\|_{L^p(\mathbb{R}^n)} ds \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \int_0^t (t+\tau-s)^{\frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \|\Delta^{-1} P \nabla \cdot f(s)\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} ds \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \int_0^t (t+\tau-s)^{\frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} ds \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \left( \int_0^{t/2} + \int_{t/2}^t \right) (t+\tau-s)^{\frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} s^{-\frac{w}{2\beta}} ds \\
& \quad \times \sup_{0<s<t} s^{\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} t^{-\frac{w}{2\beta}} \int_0^t (t+\tau-s)^{\frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} ds \times \sup_{0<s<t} s^{-\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \quad + \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} (t+\tau)^{\frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \int_0^t s^{-\frac{w}{2\beta}} ds \times \sup_{0<s<t} s^{\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} t^{-\frac{w}{2\beta}} \int_0^t (t+\tau-s)^{\frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} ds \times \sup_{0<s<t} s^{-\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \quad + \sup_{\tau>0} (t+\tau)^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta} + \frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} \int_0^t s^{-\frac{w}{2\beta}} ds \times \sup_{0<s<t} s^{\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \sup_{\tau>0} \tau^{\frac{2\beta-\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} t^{-\frac{w}{2\beta}} \int_0^t (t+\tau-s)^{\frac{-4\beta+\lceil w-(2\beta-1)+\frac{n}{p} \rceil}{2\beta}} ds \times \sup_{0<s<t} s^{-\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \quad + \sup_{\tau>0} (t+\tau)^{-1} t^{1-\frac{w}{2\beta}} \times \sup_{0<s<t} s^{\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim t^{-\frac{w}{2\beta}} \sup_{0<s<t} s^{\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)}.
\end{aligned}$$

since  $1 + n/p + w < 4\beta$ . Thus, by (iii) of Lemma 2.1 and Lemma 2.3, we have, for  $2 - 2\beta < w < 2\beta < 2$ ,

$$\begin{aligned}
& \|u(t) - e^{-t(-\Delta)^\beta} a\|_{\dot{B}_{p,q}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \int_0^t \|e^{-(t-s)(-\Delta)^\beta} \nabla \cdot f(s)\|_{\dot{B}_{p,q}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} ds \\
& \lesssim \int_0^t (t-s)^{-\frac{2-w}{2\beta}} \|f(s)\|_{\dot{B}_{p,q}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} ds \\
& \lesssim \int_0^t (t-s)^{-\frac{2-w}{2\beta}} s^{-\frac{w}{2\beta}-(1-\frac{1}{\beta})} \sup_{0<s,t} s^{\frac{w}{2\beta}+(1-\frac{1}{\beta})} \|f(s)\|_{\dot{B}_{n,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} ds \\
& \lesssim \sup_{0<s,t} s^{\frac{w}{2\beta}+1-\frac{1}{\beta}} \|f(s)\|_{\dot{B}_{n,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)}.
\end{aligned}$$

Combining the previous estimates together, we get

$$t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \lesssim t^{\frac{w}{2\beta}} \|e^{-t(-\Delta)^\beta} a\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w}{2\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)},$$

and

$$\|u(t)\|_{\dot{B}_{p,q}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \lesssim \|e^{-t(-\Delta)^\beta} a\|_{\dot{B}_{p,q}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w}{2\beta}+1-\frac{1}{\beta}} \|f(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)}.$$

Thus we can get our estimates by applying Lemma 2.3 and inequality (3.1).

#### 4. PROOF OF THEOREM 1.4

Define

$$X = \left\{ u : [0, \infty) \longrightarrow G_n^{-(2\beta-1)}(\mathbb{R}^n) \mid \nabla \cdot u = 0, \|u\|_X < \infty \right\}$$

with

$$\|u\|_X = \sup_{t > 0} \left( \|u(t)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \right).$$

Set

$$T(u)(t) = e^{-t(-\Delta)^\beta} a - \int_0^t e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u(s) \otimes v(s)) ds.$$

We want to show that  $T$  is a contraction mapping from a ball of  $X$  to itself. The case of  $p = \infty$  in Lemma 2.7 implies that

$$t^{\frac{w-2\beta}{2\beta}} \|v(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} + t^{\frac{2\beta-1}{2\beta}} \|v(t)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|v(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|v(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)}.$$

Then, according to Proposition 1.3 and Lemma 2.4, we have

$$\begin{aligned} & t^{\frac{w}{2\beta}} \|(Tu)(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \\ & \lesssim \|a\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w}{2\beta}} \|u(s) \otimes v(s)\|_{\dot{B}_{\infty,\infty}^{w-2\beta}(\mathbb{R}^n)} \\ & \lesssim \|a\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{2\beta-1}{2\beta}} \|u(s) \otimes v(s)\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w+2\beta-1}{2\beta}} \|u(s) \otimes v(s)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \\ & \lesssim \|a\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{2\beta-1}{2\beta}} \|u(s) \otimes v(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w+2\beta-1}{2\beta}} \|u(s) \otimes v(s)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \\ & \lesssim \|a\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{2\beta-1}{2\beta}} \|u(s)\|_{L^\infty(\mathbb{R}^n)} \|v(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \\ & \quad + \sup_{0 < s < t} s^{\frac{w+2\beta-1}{2\beta}} \|u(s)\|_{L^\infty(\mathbb{R}^n)} \|u(s)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \\ & \lesssim \|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|u\|_X^2. \end{aligned}$$

On the other hand, Lemma 2.8 implies that

$$\begin{aligned} \|(Tu)(t)\|_{G_n^{w-(2\beta-1)}(\mathbb{R}^n)} & \lesssim \sup_{s > 0} s^{\frac{2\beta-1}{2\beta}} \|e^{-t(-\Delta)^\beta} e^{-s(-\Delta)^\beta} a\|_{L^\infty(\mathbb{R}^n)} \\ & \quad + \int_0^t \|e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u(s) \otimes u(s))\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} ds \\ & \lesssim \|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \int_0^t (t-s)^{\frac{2\beta-1}{2\beta}} \|u(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \|u(s)\|_{L^\infty(\mathbb{R}^n)} ds \\ & \lesssim \|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|u\|_X^2. \end{aligned}$$

Hence, we get

$$\|Tu(t)\|_X \lesssim \|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|u\|_X^2$$

and

$$\|Tu - Tv\|_X \lesssim (\|u\|_X + \|v\|_X)\|u - v\|_X.$$

Therefore, the contraction mapping principle implies there exists a unique solution to equations (1.1) if  $\|a\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)}$  is small enough.

## 5. PROOF OF THEOREM 1.5

The solution  $\{u, b\}$  to equations (1.7) can be written as

$$u(t, x) = e^{-t(-\Delta)^\beta} u_0(x) - B(u, u) + B(u, u) := F_1(u, b),$$

$$b(t, x) = e^{-t(-\Delta)^\beta} b_0(x) - B(u, b) + B(b, u) := F_2(u, b),$$

with

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u \otimes v)(s) ds.$$

Define

$$Y = \left\{ (u, b) : (0, \infty) \longrightarrow G_n^{-(2\beta-1)}(\mathbb{R}^n) \mid \nabla \cdot u = \nabla \cdot b = 0, \|(u, b)\|_Y < \infty \right\}$$

with

$$\|(u, b)\|_Y = \sup_{t>0} \left( \|(u, b)(t)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + t^{\frac{w-1}{2\beta}} \|(u, b)(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \right) < \infty,$$

$$\|(u, b)\|_Y = \|u\|_Y + \|b\|_Y.$$

We want to show that  $F_1$  and  $F_2$  are contraction mappings from a ball of  $Y$  to itself. We rewrite the solution  $(u, b)$  as

$$\begin{pmatrix} u \\ b \end{pmatrix} = \begin{pmatrix} F_1(u, b) \\ F_2(u, b) \end{pmatrix} := F(u, b).$$

Then we have

$$\begin{aligned} & t^{\frac{w}{2\beta}} \|F_1(u, b)(t)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \\ & \lesssim \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0<s<t} s^{\frac{w}{2\beta}} \|(u \otimes u, b \otimes b)(s)\|_{\dot{B}_{\infty,\infty}^{w-2\beta}(\mathbb{R}^n)} \\ & \lesssim \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0<s<t} s^{\frac{2\beta-1}{2\beta}} \|(u \otimes u, b \otimes b)(s)\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} \\ & \quad + \sup_{0<s<t} s^{\frac{w+2\beta-1}{2\beta}} \|(u \otimes u, b \otimes b)(s)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \\ & \lesssim \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0<s<t} s^{\frac{2\beta-1}{2\beta}} \|(u \otimes u, b \otimes b)(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \\ & \quad + \sup_{0<s<t} s^{\frac{w+2\beta-1}{2\beta}} \|(u, b)(s)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \|(u, b)(s)\|_{L^\infty(\mathbb{R}^n)} \\ & \lesssim \|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{0<s<t} s^{\frac{2\beta-1}{2\beta}} \|(u, b)(s)\|_{L^\infty} \|(u, b)(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \\ & \quad + \sup_{0<s<t} s^{\frac{w+2\beta-1}{2\beta}} \|(u, b)(s)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \|(u, b)(s)\|_{L^\infty(\mathbb{R}^n)} \\ & \lesssim \|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \\ & \quad + \sup_{0<s<t} \left[ s^{\frac{2\beta-1}{2\beta}} \|(u, b)(s)\|_{L^\infty} \left( \|(u, b)(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + s^{\frac{w}{2\beta}} \|(u, b)(s)\|_{\dot{B}_{\infty,\infty}^{w-(2\beta-1)}(\mathbb{R}^n)} \right) \right] \\ & \lesssim \|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|(u, b)\|_Y^2. \end{aligned}$$

Similarly, we get

$$\begin{aligned}
& \|F_1(u, b)(t)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \\
& \lesssim \sup_{s>0} s^{\frac{2\beta-1}{2\beta}} \|e^{-t(-\Delta)^\beta} e^{-s(-\Delta)^\beta} |u_0|(s)\|_{L^\infty(\mathbb{R}^n)} \\
& \quad + \int_0^t \|e^{-(t-s)(-\Delta)^\beta} P\nabla \cdot (u \otimes u)(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} ds \\
& \quad + \int_0^t \|e^{-(t-s)(-\Delta)^\beta} P\nabla \cdot (b \otimes b)(s)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} ds \\
& \lesssim \|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \int_0^t (t-s)^{\frac{2\beta-1}{2\beta}} \|(u, b)\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} \|(u, b)(s)\|_{L^\infty(\mathbb{R}^n)} ds \\
& \lesssim \|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|(u, b)\|_Y^2.
\end{aligned}$$

Thus, we have

$$\|F_1(u, b)\|_Y \lesssim \|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|(u, b)\|_Y^2$$

and

$$\|F_1(u, b)(t) - F_1(u', b')(t)\|_Y \lesssim \|(u - u', b - b')\|_Y (\|(u, b)\|_Y + \|(u', b')\|_Y).$$

Similarly, we can prove that

$$\|F_2(u, b)\|_Y \lesssim \|u_0\|_{G_n^{-(2\beta-1)}(\mathbb{R}^n)} + \|u\|_Y \|b\|_Y$$

and

$$\|F_2(u, b)(t) - F_2(u', b')(t)\|_Y \lesssim \|(u - u', b - b')\|_Y (\|(u, b)\|_Y + \|(u', b')\|_Y).$$

These estimates imply that

$$\|F(u, b) - F(u', b')\|_Y \lesssim \|(u - u', b - b')\|_Y (\|(u, b)\|_Y + \|(u', b')\|_Y).$$

Therefore, the contraction mapping principle finishes the proof.

## 6. PROOF OF PROPOSITION 1.6

Define

$$K = \left\{ f \in L^\infty \left( (0, \infty); \dot{B}_{p, \infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n) \right) : \nabla \cdot f = 0, \|f\|_K < \infty \right\}$$

with

$$\|f\|_K = \sup_{t>0} \left( \|u(t)\|_{\dot{B}_{p, \infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|u(t)\|_{\dot{B}_{p, \infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \right).$$

Let

$$Tu(t) = e^{-t(-\Delta)^\beta} a - \int_0^t e^{-(t-s)(-\Delta)^\beta} P\nabla \cdot (u(s) \otimes u(s)) ds.$$

We want to prove that  $T$  is a contraction mapping from a ball of  $K$  to itself. It follows from Lemmas 2.4 and 2.7 that

$$\begin{aligned}
& \|Tu(t)\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \|Tu(t)\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \|a\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + t^{\frac{w}{2\beta}} \int_0^t \|e^{-(t-s)(-\Delta)^\beta} P\nabla \cdot (u(s) \otimes u(s))\|_{\dot{B}_{p,\infty}^{w-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} ds \\
& \lesssim \|a\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w}{2\beta}} \|(u(s) \otimes u(s))\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \|a\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \sup_{0 < s < t} s^{\frac{w}{2\beta}} \|u(s)\|_{L^\infty(\mathbb{R}^n)} \|u(s)\|_{\dot{B}_{p,\infty}^{w-2\beta+\frac{n}{p}}(\mathbb{R}^n)} \\
& \lesssim \|a\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)} + \|u\|_K^2
\end{aligned}$$

since  $0 < w - 2\beta + \frac{n}{p} < 1$ ,  $n \leq p < \infty$ . Similarly, we get

$$\|Tu - Tv\|_K \lesssim (\|u\|_K + \|v\|_K) \|u - v\|_K, \quad \text{for } u, v \in K.$$

Thus, these estimates imply that  $T$  is a contraction mapping for  $\|a\|_{\dot{B}_{p,\infty}^{-(2\beta-1)+\frac{n}{p}}(\mathbb{R}^n)}$  small enough. Therefore, we can finish the proof by the contraction mapping principle.

## 7. PROOF OF PROPOSITION 1.12

*Proof.* We can write

$$e^{-t(-\Delta)^\beta} f = e^{-(t-u)(-\Delta)^\beta} e^{-u(-\Delta)^\beta} f$$

and

$$e^{-t(-\Delta)^\beta} f(x) = \frac{2}{t} \int_0^{t/2} e^{-(t-u)(-\Delta)^\beta} e^{-u(-\Delta)^\beta} f ds.$$

According to the definition of  $e^{-t(-\Delta)^\beta}$ , it is a convolution operator with a positive Kernel  $K_t^\beta(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi - t|\xi|^{2\beta}} d\xi$  satisfying  $K_t^\beta(x) = \frac{1}{t^{n/2\beta}} K_1^\beta(\frac{x}{t^{1/2\beta}})$ . Then, using Hölder's inequality, we obtain that

$$\begin{aligned}
& |e^{-t(-\Delta)^\beta} f(x_0)| = \left| \frac{2}{t} \int_0^{t/2} \int_{\mathbb{R}^n} K_{t-u}^\beta(x - x_0) e^{-u(-\Delta)^\beta} f(x_0) dx ds \right| \\
& \lesssim \frac{2}{t} \int_0^{t/2} \left( \int_{\mathbb{R}^n} K_{t-u}^\beta(x - x_0) |e^{-u(-\Delta)^\beta} f(x_0)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |K_{t-u}^\beta(x - x_0)| dx \right)^{1/2} ds \\
& \lesssim \frac{2}{t} \int_0^{t/2} \left( \int_{\mathbb{R}^n} K_{t-u}^\beta(x - x_0) |e^{-u(-\Delta)^\beta} f(x_0)|^2 dx \right)^{1/2} ds \\
& \lesssim \left( \frac{2}{t^{\frac{\zeta-1}{\beta}}} \int_0^{t/2} \int_{\mathbb{R}^n} K_{t-u}^\beta(x - x_0) u^{\frac{\zeta-1}{\beta}} |e^{-u(-\Delta)^\beta} f(x_0)|^2 dx ds \right)^{1/2}.
\end{aligned}$$

By Miao, Yuan and Zhang's [18, Lemma 2.1], we have

$$K_{t-u}^\beta(x - x_0) \lesssim \frac{1}{(t-u)^{n/2\beta}} \frac{1}{\left(1 + \frac{|x-x_0|}{(t-u)^{1/2\beta}}\right)^{n+2\beta}}.$$

Thus

$$\begin{aligned}
I &= \int_{\mathbb{R}^n} K_{t-u}^\beta(x-x_0) u^{\frac{\zeta-1}{\beta}} |e^{-u(-\Delta)^\beta} f(x)|^2 dx \\
&\lesssim \int_{\mathbb{R}^n} \frac{1}{(t-u)^{n/2\beta}} \frac{1}{\left(1 + \frac{|x-x_0|}{(t-u)^{1/2\beta}}\right)^{n+2\beta}} u^{\frac{\zeta-1}{\beta}} |e^{-u(-\Delta)^\beta} f(x)|^2 dx \\
&\lesssim \int_{\frac{x-x_0}{(t-u)^{\frac{1}{2\beta}}} \in k+[0,1]^n} \frac{1}{(t-u)^{n/2\beta}} \frac{1}{\left(1 + \frac{|x-x_0|}{(t-u)^{1/2\beta}}\right)^{n+2\beta}} u^{\frac{\zeta-1}{\beta}} |e^{-u(-\Delta)^\beta} f(x)|^2 dx.
\end{aligned}$$

Since  $0 < u < \frac{t}{2}$  and  $\frac{t}{2} < t-u < t$ , we can get

$$I \lesssim \frac{1}{t^{n/2\beta}} \int_0^{t/2} \int_{|x-x_0| \leq t^{\frac{1}{2\beta}}} u^{\frac{\zeta-1}{\beta}} |e^{-u(-\Delta)^\beta} f(x)|^2 dx.$$

This gives

$$\begin{aligned}
\|e^{-t^{2\beta}(-\Delta)^\beta} f\|_{L^\infty} &\lesssim \left( \frac{2}{t^{\frac{\zeta-1}{\beta}}} \frac{1}{t^n} \int_0^{t/2} \int_{|x-x_0| \leq t^{\frac{1}{2\beta}}} s^{\frac{\zeta-1}{\beta}} |e^{-s(-\Delta)^\beta} f(x)|^2 dx ds \right)^{1/2} \\
&\lesssim \left( \frac{2}{t^{\frac{\zeta-1}{\beta}}} t^{-n} \int_0^{t/2} \int_{|x-x_0| \leq t^{\frac{1}{2\beta}}} s^{\frac{\zeta-1}{\beta}} |e^{-s(-\Delta)^\beta} f(x)|^2 dx ds \right)^{1/2},
\end{aligned}$$

that is,

$$t^\zeta \|e^{-t^{2\beta}(-\Delta)^\beta} f\|_{L^\infty} \lesssim \left( t^{-n} \int_0^{t/2} \int_{|x-x_0| \leq t^{\frac{1}{2\beta}}} s^{\frac{\zeta-1}{\beta}} |e^{-s(-\Delta)^\beta} f(x)|^2 dx ds \right)^{1/2}.$$

It follows from Miao, Yuan and Zhang's [18, Proposition 2.1] that, for  $s < 0$ ,  $f \in \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$  if and only if

$$\sup_{r>0} r^{-s/2\beta} \|e^{-r(-\Delta)^\beta} f\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Thus, the previous estimate implies that  $BMO^{-\zeta}(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^{-\zeta}(\mathbb{R}^n)$ .  $\square$

## 8. PROOF OF PROPOSITION 1.9

We need the following lemma which can be proved easily.

**Lemma 8.1.** *For  $\zeta \geq 0$ ,  $(-\Delta)^{\zeta/2} e^{-(\Delta)^\beta}$  is a convolution operator with kernel  $K^{\zeta,\beta}(x) \in L^1(\mathbb{R}^n)$ .*

We divide the proof into two parts. First, we prove that  $f \in BMO^{-\zeta}(\mathbb{R}^n)$  under the assumption of the existence of a distribution  $g \in BMO(\mathbb{R}^n)$  with  $f = (-\Delta)^{\zeta/2} g$ . From this assumption, we have, for all  $s > 0$ ,

$$s^{\zeta/\beta} |e^{-s(-\Delta)^\beta} (-\Delta)^{\frac{\zeta}{2}} g|^2 = |K_{\zeta,s^{\frac{1}{2\beta}}} * g|^2$$

with

$$K_{\zeta,s^{\frac{1}{2\beta}}}(x) = s^{-\frac{n}{2\beta}} K_\zeta\left(\frac{x}{s^{\frac{1}{2\beta}}}\right).$$

Here  $K_\zeta \in L^1(\mathbb{R}^n)$  and

$$\widehat{K_{\zeta,s^{\frac{1}{2\beta}}}}(\xi) = \widehat{K_\zeta}(s^{\frac{1}{2\beta}} \xi) = s^{\frac{\zeta}{2\beta}} |\xi|^\zeta e^{-s|\xi|^{2\beta}}.$$

Thus

$$\int_{\mathbb{R}^n} K_\zeta(x) ds = 0 \text{ and } |K_\zeta(x)| \lesssim \frac{1}{(1+|x|)^{n+\zeta}}.$$

For more about the kernel of  $e^{-t(-\Delta)^\beta}$ , see Miao, Yuan and Zhang [18]. Then we have

$$\begin{aligned} \sup_{\xi \in \mathbb{R}^n} \int_0^\infty |\widehat{K}_\zeta(t\xi)|^2 \frac{dt}{t} &= \sup_{|\xi|=1} \int_0^\infty |\widehat{K}_\zeta(t\xi)|^2 \frac{dt}{t} \\ &= \sup_{|\xi|=1} \int_0^\infty (t^\zeta e^{-t^{2\beta}})^2 \frac{dt}{t} \\ &= \int_0^\infty t^{2\zeta-1} e^{-2t^{2\beta}} dt \\ &= \frac{2^{-\frac{\zeta}{\beta}} - 1}{\beta} \int_0^\infty t^{\frac{\zeta}{\beta}-1} e^{-t} dt \\ &= \frac{2^{-\frac{\zeta}{\beta}} - 1}{\beta} \Gamma\left(\frac{\zeta}{\beta}\right) < \infty, \end{aligned}$$

since  $\frac{\zeta}{\beta} > 0$ . So  $d\mu(x, s) = |K_{\zeta, s^{\frac{1}{2\beta}}} * g|^2 \frac{ds dx}{s}$  is a Carleson measure and

$$\int \int_{0 < s < t, |x-x_0| < t^{\frac{1}{2\beta}}} |(K_{\zeta, s^{\frac{1}{2\beta}}} * g)(x)|^2 \frac{ds dx}{s} \leq C \|g\|_{BMO(\mathbb{R}^n)}^2 t^{\frac{n}{2\beta}}.$$

That is  $\|f\|_{BMO^{-\zeta}(\mathbb{R}^n)} \leq C \|g\|_{BMO(\mathbb{R}^n)}$ .

Second, we prove the existence of  $g \in BMO(\mathbb{R}^n)$  with  $f = (-\Delta)^{\frac{\zeta}{2}} g$  when  $f \in BMO^{-\zeta}(\mathbb{R}^n)$ . Proposition 1.8 implies that we can get

$$g = \sum_{j < 0} g_j - g_j(0) + \sum_{j > 0} g_j$$

with  $g_j = \Delta_j g$  such that  $f = (-\Delta)^{\zeta/2} g$  and  $g \in \dot{B}_{\infty, \infty}^0(\mathbb{R}^n)$ . In fact,

$$\widehat{g}(\xi) = \sum_{j < 0} \widehat{g}_j(\xi) - \widehat{g}_0(\xi) + \sum_{j > 0} \widehat{g}_j(\xi),$$

and

$$\begin{aligned} |\xi|^\zeta \widehat{g}(\xi) &= \sum_{j < 0} |\xi|^\zeta \widehat{g}_j(\xi) - |\xi|^\zeta \widehat{g}_0(\xi) + \sum_{j > 0} |\xi|^\zeta \widehat{g}_j(\xi) \\ &= |\xi|^\zeta \sum_{j \in \mathbb{Z}} \widehat{g}_j(\xi) = \sum_{j \in \mathbb{Z}} \widehat{\Delta_j(f)}(\xi) = \widehat{f}(\xi), \end{aligned}$$

according to the homogeneous Littlewood-Paley decomposition of  $f$ . On the other hand, to see  $g \in \dot{B}_{\infty, \infty}^0(\mathbb{R}^n)$ , we have,

$$g_j = \Delta_j g = \Delta_j (-\Delta)^{\zeta/2} f,$$

and

$$\begin{aligned} \widehat{g}_j(\xi) &= |\xi|^{-\zeta} \phi(2^{-j}\xi) \widehat{f}(\xi) \\ &= 2^{-j\zeta} |2^{-j}\xi|^{-\zeta} \phi(2^{-j}\xi) \widehat{f}(\xi) \\ &= 2^{-jr} h_j(\xi) |2^{-j}\xi|^{-\zeta} \phi(2^{-j}\xi) \widehat{f}(\xi). \end{aligned}$$

Here  $h_j \in C_0^\infty(\mathbb{R}^n)$  satisfying  $h_j = 1$  on  $C_j$  and  $\text{supp}(h_j) \subset 2C_j$ . Let

$$g_j = 2^{-j\zeta} \Delta_j f * (h_j |2^{-j}\xi|^{-\zeta})^\vee,$$

where  $(h_j |2^{-j}\xi|^{-\zeta})^\vee \in L^\infty(\mathbb{R}^n)$ . It follows from  $h_j |2^{-j}\xi|^{-\zeta} \in l^\infty(\mathbb{Z})$  that  $\|\Delta_j g\|_{L^\infty(\mathbb{R}^n)} \in l^\infty(\mathbb{Z})$ .

We need to prove that  $g \in BMO(\mathbb{R}^n)$ . In fact, let  $\eta$  by

$$\widehat{\eta}(s^{\frac{1}{2\beta}} \xi) = |s^{\frac{1}{2\beta}} \xi|^\zeta e^{-s|\xi|^{2\beta}}.$$

So

$$\widehat{\eta_{s^{\frac{1}{2\beta}} * g}}(\xi) = \widehat{\eta}(s^{\frac{1}{2\beta}} \xi) \widehat{g}(\xi) = |s^{\frac{1}{2\beta}} \xi|^\zeta e^{-s|\xi|^{2\beta}} \widehat{g}(\xi)$$

and

$$\widehat{g}(\xi) = \sum_{j \in \mathbb{Z}} |\xi|^{-\zeta} \widehat{\Delta_j}(f)(\xi) = |\xi|^{-\zeta} \widehat{f}(\xi).$$

This tells us

$$\widehat{\eta_{s^{\frac{1}{2\beta}} * g}}(\xi) = s^{\frac{\zeta}{2\beta}} e^{-s|\xi|^{2\beta}} \widehat{f}(\xi).$$

So

$$\eta_{s^{\frac{1}{2\beta}}} * g(x) = s^{\frac{\zeta}{2\beta}} e^{-s(-\Delta)^\beta} f(x).$$

It follows from  $f \in BMO^{-\zeta}(\mathbb{R}^n)$  and  $\eta$  satisfying the assumptions of Lemma 2.6 that

$$\begin{aligned} & |B(x_0, t^{\frac{1}{2\beta}})|^{-1} \int_0^t \int_{|x-x_0| < t^{\frac{1}{2\beta}}} |\eta_{s^{\frac{1}{2\beta}}} * g|^2 \frac{dsdx}{s} \\ &= t^{\frac{n}{2\beta}} \int_{0 < s < t} \int_{|x-x_0| < t^{\frac{1}{2\beta}}} |s^{\frac{\zeta}{2\beta}} e^{-s(-\Delta)^\beta} f(x)|^2 \frac{dsdx}{s} \\ &\lesssim \sup_{t>0} \sup_{x_0 \in \mathbb{R}^n} \left( t^{-n} \int_0^t \int_{|x-x_0| < t} s^{\frac{\zeta-1}{\beta}} |e^{-s(-\Delta)^\beta} f(x)|^2 dsdx \right) \\ &= \|f\|_{BMO^{-\zeta}(\mathbb{R}^n)}^2. \end{aligned}$$

The previous estimate and Lemma 2.6 imply that  $g \in BMO(\mathbb{R}^n)$ . This finishes the proof.

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